Pergamon

Int. J. Heat Mass Transfer. Vol. 41, No. 1, pp. 241–243, 1998

Printed in Great Britain
0017–9310/98 $19.00+0.00

PII: S0017–9310(97)00103–8

TECHNICAL NOTES

Convection in a porous medium with inclined temperature gradient and vertical throughflow

D. A. NIELD
Department of Engineering Science, University of Auckland, Private Bag 92109, Auckland, New Zealand

(*Received* 26 *February* 1996 *and in final form* 28 *March* 1997)

1. INTRODUCTION

The present study is of relevance to the performance of packed bed reactors [1], but the author's primary motivation is to investigate a scientific situation which has not been studied previously. Motivated by the belief that it serves as a paradigm for convection induced by an inclined applied temperature gradient in general situations, the author [2, 3] has studied the case of such convection in a shallow layer of a saturated porous medium. The horizontal component of the applied gradient induces a Hadley circulation which, in the central portion of the flow, is approximately independent of horizontal position and can be treated as a uniform flow. This flow becomes unstable when the vertical component of the applied gradient is sufficiently great.

The present paper is an extension, to the case where vertical throughflow is present, of refs [2] and [3]. It is also an extension, to the case of inclined rather than vertical gradients, of the studies of effect of throughflow, on the onset of convection in a porous medium layer, by Jones and Persichetti [1] and Nield [4]. As far as the author is aware, these are the only published papers on the onset of convection with vertical throughflow in a porous medium. Other aspects of convection induced by inclined gradients in a porous medium have been reviewed by Lage and Nield [5].

2. BASIC EQUATIONS AND STEADY-STATE SOLUTION

Cartesian axes are chosen with the z^*-axis vertically upwards and the x^*-axis in the direction of the applied horizontal temperature gradient β. The superscript asterisks denote dimensional variables. The porous medium occupies a layer of height H. The vertical temperature difference across the boundaries is ΔT (see Fig. 1). It is assumed that the Oberbeck–Boussinesq approximation is valid, and that flow in the porous medium is governed by Darcy's law. Accordingly, the governing equations are

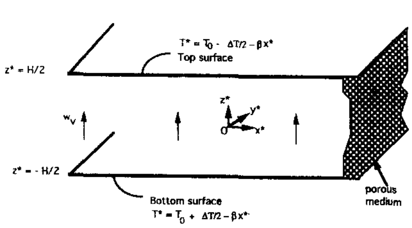

Fig. 1. Definition sketch for the problem.

$$\nabla^* \cdot \mathbf{v}^* = 0 \quad (1)$$

$$0 = -\nabla^* P^* - (\mu/K)\mathbf{v}^* + \rho_f^* \mathbf{g} \quad (2)$$

$$(\rho c)_m(\partial T^*/\partial t^*) + (\rho c_p)_f \mathbf{v}^* \cdot \nabla^* T^* = k_m \nabla^{*2} T^* \quad (3)$$

$$\rho_f^* = \rho_0[1 - \gamma_T(T^* - T_0)]. \quad (4)$$

Here $(u^*, v^*, w^*) = \mathbf{v}^*$, P^* and T^* are the seepage (Darcy) velocity, pressure and temperature, respectively. The subscripts m and f refer to the porous medium and the fluid, respectively. Also, μ, ρ and c denote viscosity, density and specific heat, while K is the permeability of the medium, k_m is the effective thermal conductivity, and γ_T is the thermal expansion coefficient.

The throughflow velocity is denoted by w_V, so the boundary conditions are

$$w^* = w_v$$
$$T^* = T_0 - (\pm \Delta T)/2 - \beta x^* \quad \text{at } z^* = \pm H/2. \quad (5)$$

We define non-dimensional quantities by

$$\mathbf{x} = \mathbf{x}^*/H \quad t = \alpha_m t^*/AH^2 \quad (u, v, w) = \mathbf{v} = H\mathbf{v}^*/\alpha_m$$
$$P = K(P^* + \rho_0 g z^*)/\mu\alpha_m \quad T = R_v(T^* - T_0)/\Delta T$$

where $\alpha_m = k_m/(\rho c_p)_f$ and $A = (\rho c_m)/(\rho c_p)_f$. The non-dimensional parameters which arise are the throughflow Péclet number

$$Q_v = w_v H/\alpha_m \quad (6)$$

the vertical Rayleigh number

$$R_v = \rho_0 g\gamma KH\Delta T/\mu\alpha_m \quad (7)$$

and the horizontal Rayleigh number

$$R_h = \rho_0 g\gamma KH^2\beta/\mu\alpha_m. \quad (8)$$

The governing equations now take the form

$$\nabla \cdot \mathbf{v} = 0 \quad (9)$$

$$0 = -\nabla P - \mathbf{v} + T\mathbf{k} \quad (10)$$

$$\partial T/\partial t + \mathbf{v} \cdot \nabla T = \nabla^2 T. \quad (11)$$

The boundary conditions are now

$$w = Q_v \quad T = -(\pm R_v)/2 - R_h x \quad \text{at } z = \pm 1/2. \quad (12)$$

Equations (9)–(12) have a steady-state solution of the form

$$T_s = \tilde{T}(z) - R_h x \quad u_s = U(z) \quad v_s = 0 \quad w_s = Q_v \tag{13}$$
$$P_s = P(x, y, z).$$

This is a solution provided that

$$DU = R_h \tag{14}$$

$$D^2\tilde{T} - Q_v D\tilde{T} = -R_h U. \tag{15}$$

Here D denotes the derivative operator d/dz. It is assumed that there is no net horizontal flow so $\langle U \rangle = 0$. Here the angle brackets denote an average with respect to the vertical coordinate, which is equivalent to an integral with respect to z from $-1/2$ to $1/2$.

The steady-state solution is thus

$$U_s = R_h z \tag{16}$$

$$\tilde{T}_s = \frac{R_h^2}{Q_v}\left[\frac{z^2}{2} - \frac{1}{8}\right] + \frac{R_h^2 z}{Q_v^2} - \frac{Q_v^2 R_h + R_h^2}{2Q_v^2 \sinh(Q_v/2)}[\exp(Q_v z) - \cosh(Q_v/2)]. \tag{17}$$

3. STABILITY ANALYSIS

We now perturb the steady-state solution. We write $\mathbf{v} = \mathbf{v}_s + \mathbf{v}'$, $T = T_s + \theta'$, $P = P_s + p'$. The linearized perturbation equations are

$$\nabla \cdot \mathbf{v}' = 0 \tag{18}$$

$$\nabla p' + \mathbf{v}' - \theta' \mathbf{k} = 0 \tag{19}$$

$$\partial\theta'/\partial t + U\partial\theta'/\partial x + Q_v\, \partial\theta'/\partial z - R_h u' + (D\tilde{T})w' = \nabla^2\theta'. \tag{20}$$

We make the normal mode expansion

$$[u', v', w', \theta', p'] = [u(z), v(z), w(z), \theta(z), p(z)] \times \exp\{i(kx + ly - \sigma t)\}. \tag{21}$$

We substitute this into the perturbation equations and eliminate p, u, and v from the resulting equations to obtain

$$(D^2 - \alpha^2)w + \alpha^2\theta = 0 \tag{22}$$

$$(D^2 - \alpha^2 + i\sigma - ikU)\theta + i\alpha^{-2}kR_h Dw - (D\tilde{T})w - Q_v D\theta = 0 \tag{23}$$

where $\alpha = (k^2 + l^2)^{1/2}$ is the overall horizontal wavenumber. We refer to a disturbance with $k = 0$ as a longitudinal mode and one with $l = 0$ as a transverse mode.

The last two equations must be solved subject to appropriate boundary conditions. For the case of boundaries at which the perturbation velocity and temperature are zero, we have

$$w = \theta = 0 \quad \text{at}\ z = \pm\frac{1}{2}. \tag{24}$$

The problem is now reduced to that of solving equations (22)–(24), where U is given by equation (16) and

$$D\tilde{T} = R_h^2 z/Q_v + R_h^2/Q_v^2 - \frac{Q_v^2 R_h + R_h^2}{2Q_v \sinh(Q_v/2)}\exp(Q_v z). \tag{25}$$

Without loss of generality we may regard R_v as the eigenvalue with R_h, Q_v, σ, k and l as parameters. At neutral stability σ has to be real and chosen so that R_v is real. Subject to this constraint, the critical value of R_v is its minimum, as σ, k and l are varied.

To solve the differential equation system, a Galerkin approximation of order N, as in Ref. [2], was used. We select as trial functions (satisfying the boundary conditions)

$$w_{2p-1} = \theta_{2p-1} = \cos(2p-1)\pi z$$
$$w_{2p} = \theta_{2p} = \sin 2p\pi z \quad \text{for}\ p = 1, 2, \ldots, [(N+1)/2]$$

where the square brackets denote 'integer part of'.

The standard procedure [2, 3] leads to the eigenvalue equation in the form

$$\det(A_{ij}) = 0 \tag{26}$$

where, for m, $n = 1, 2$

$$A_{2m-1,2n-1} = \langle Dw_m Dw_n + \alpha^2 w_m w_n \rangle$$
$$A_{2m-1,2n} = -\alpha^2 \langle w_m \theta_n \rangle$$
$$\alpha_{2m,2n-1} = \langle D\tilde{T}\theta_m w_n - i\alpha^{-2}kR_h\theta_m Dw_n \rangle$$
$$A_{2m,2n} = \langle D\theta_m D\theta_n + (\alpha^2 - i[\sigma - kU])\theta_m\theta_n + Q_v\theta_m D\theta_m \rangle.$$

The various integrals involved are easily evaluated. One finds, for example, that

$$\langle w_m w_n \rangle = \frac{1}{2}\delta_{mn} \quad \langle Dw_m Dw_n \rangle = \frac{1}{2}m^2\pi^2\delta_{mn}$$

where δ_{mn} is the Kronecker delta

$$\langle \theta_m D\theta_n \rangle = \frac{2mn\nu_{mn}}{n^2 - m^2} \quad \langle z\theta_m\theta_n \rangle = \frac{4mn\nu_{mn}}{\pi^2(m^2 - n^2)^2}$$

where

$$\nu_{mn} = \begin{cases} 0 & \text{if}\ m+n\ \text{is even} \\ 1 & \text{if}\ (m+n+1)/2\ \text{is even} \\ -1 & \text{if}\ (m+n+1)/2\ \text{is odd} \end{cases}$$

$$\langle e^{\lambda z}\theta_m w_n \rangle = \begin{cases} \Phi\sinh(\lambda/2) & \text{if}\ m+n\ \text{is even and}\ \left[\frac{m+1}{2}\right] + \left[\frac{n+1}{2}\right]\ \text{is even} \\ -\Phi\sinh(\lambda/2) & \text{if}\ m+n\ \text{is even and}\ \left[\frac{m+1}{2}\right] + \left[\frac{n+1}{2}\right]\ \text{is odd} \\ \Phi\cosh(\lambda/2) & \text{if}\ m+n\ \text{is odd and}\ \left[\frac{m+1}{2}\right] + \left[\frac{n+1}{2}\right]\ \text{is even} \\ -\Phi\cosh(\lambda/2) & \text{if}\ m+n\ \text{is odd and}\ \left[\frac{m+1}{2}\right] + \left[\frac{n+1}{2}\right]\ \text{is odd} \end{cases}$$

where

$$\Phi = \frac{4mn\pi^2\lambda}{[\lambda^2 + (m+n)^2\pi^2][\lambda^2 + (m-n)^2\pi^2]}.$$

4. ANALYTIC AND NUMERICAL SOLUTIONS AND DISCUSSION

At the second-order ($N = 2$) approximation it is feasible to expand the eigenvalue equation (26) algebraically. Taking the imaginary part of the equation yields $\sigma = 0$, and the real part yields

$$R_v = \frac{(\pi^2+\alpha^2)^2}{\alpha^2}\left(1+\frac{Q_v^2}{4\pi^2}\right)+\frac{R_h^2}{4\pi^2}. \quad (27)$$

Thus the critical (i.e. minimum as α is varied) wavenumber is $\alpha_c = \pi$ and the critical vertical Rayleigh number is

$$R_{vc} = 4\pi^2 + Q_v^2 + R_h^2/4\pi^2. \quad (28)$$

For the case $Q_v = 0$, the result given in ref. [2] is recovered. Clearly the effect of vertical throughflow is stabilizing for the situation we are considering. This agrees with the results reported in refs [1] and [4].

The formula (27) is known to be exact when $Q_v = 0$, and so is expected to be a good approximation when Q_v is small, but non-zero. At order $N = 12$ a numerical method, described in ref. [3], was used to calculate the minimum value of R_v. It was shown by Nield [2] that the favoured form of the disturbance for the case $Q_v = 0$ is in the form of non-oscillatory longitudinal rolls, and accordingly the results reported in this paper are for $\sigma = 0$ and $k = 0$. A check was made that, for the range of parameters for which results are reported, the longitudinal mode is indeed the favoured one. Since the problem is not self adjoint, the eigenvalue σ is not necessarily real. However, it appears that there is no physical mechanism present which can lead to oscillatory modes being favoured, at least for the case of small Q_v.

The results for R_{vc} are presented in Table 1. (Since the eigenvalue equation is singular for $Q_v = 0$, extrapolation is needed to obtain results for that value of Q_v.) The range for R_h has been restricted so that the approximation at this order gives results accurate to within 0.1%. The values for α_c presented in Table 2 are accurate to 1%.

The values of R_{vc} presented in Table 1 differ from those predicted from equation (28) by less than 10%. The results for the case $Q_v = 0$ agree well with those reported in ref. [2] and the results for $R_h = 0$ are in line with those reported in ref. [1]. It is seen that increments between adjacent rows of the table are uniform across a row, showing that the stabilizing effects of the horizontal applied temperature gradient (represented by R_h) and the vertical throughflow (represented by Q_v) are approximately additive, for the parameter range for which the calculations have been made. We note that, for this case in which the upper and lower boundary conditions are identical, the effect of throughflow is independent of the sign of Q_v. The stabilizing effect is a result of the throughflow giving rise to a temperature distribution in which the gradient is significant only in a sub-layer of depth δ, say, and the effective Rayleigh number is that based on δ rather than H, the full layer depth. The bulk of the convection is confined to the sublayer. The applied temperature difference needed to induce convection is accordingly increased, as a result of the throughflow, by a factor of the order of H/δ. This means that the critical value of R_v is increased by that factor.

The results presented in Table 2 indicate that the critical wavenumber increases with both R_h and Q_v for the parameter range investigated. At present there are no experimental results available for comparison. When they are available it may be desirable to extend the parameter range for calculations.

Table 1. Values of the critical vertical Rayleigh number R_{vc} for various values of Q_V and R_H

	$R_H = 0$	10	20	30	40
$Q_v = 0$	39.48	42.01	49.56	62.01	79.02
1	40.88	43.40	50.94	63.34	80.31
2	45.08	47.60	55.13	67.49	84.38
3	52.07	54.58	62.08	74.38	91.12
4	61.67	64.16	71.61	83.78	100.3
5	73.40	75.87	83.17	95.06	111.0
6	86.59	88.97	95.98	107.3	122.0
7	100.5	120.8	109.3	119.6	132.8
8	114.7	116.8	122.7	131.9	143.7

Table 2. Values of the critical horizontal wavenumber α_c for various values of Q_V and R_H

	$R_H = 0$	10	20	30	40
$Q_V = 0$	3.14	3.14	3.15	3.16	3.20
1	3.18	3.18	3.18	3.20	3.25
2	3.29	3.29	3.29	3.21	3.37
3	3.49	3.49	3.50	3.52	3.58
4	3.79	3.79	3.81	3.85	3.95
5	4.20	4.21	4.25	4.35	4.58
6	4.73	4.76	4.86	5.06	5.52
7	5.38	5.43	5.58	5.89	6.45
8	6.10	6.16	6.35	6.71	7.28

REFERENCES

1. Jones, M. C. and Persichetti, J. M., Convective instability in packed beds with throughflow. *AIChE Journal*, 1986, **32,** 1555–1557.
2. Nield, D. A., Convection in a porous medium with inclined temperature gradient. *International Journal of Heat and Mass Transfer*, 1991, **34,** 87–92.
3. Nield, D. A., Convection in a porous medium with inclined temperature gradient: additional results. *International Journal of Heat and Mass Transfer*, 1994, **37,** 3021–3025.
4. Nield, D. A., Convective instability in porous media with throughflow. *AIChE Journal*, 1987, **33,** 1222–1224.
5. Lage, J. L. and Nield, D. A., Convection induced by inclined gradients in a shallow porous medium layer. *Journal of Porous Media* (to appear).